# My Mathematical Engravings

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**Abstract.** The work of an engraver is shown through the presentation of three types of engravings concerning minimal surfaces, closed surfaces without singularities, and bi-periodic functions.

## 1 Introduction: The Engraver's Job.

I make engravings on copper. From the following images, one can follow some of the many steps that end in the production of an image from the engraving. In this case it is an engraving of an olive tree that could be used as an emblem for Provence, the region where I live and which I try to honor with some of my works.

**Fig.1.**

**Fig. 2.**

**Fig. 3.**

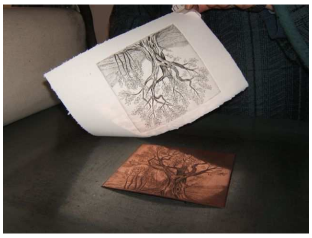

**Fig. 4.** **Fig. 5.** **Fig. 6.**

## Engraving

Engraving: The engraving method consists of cutting incisions on a copper plate with a kind of chisel called a burin. The finished engraving is then printed using the intaglio method. That is, the entire surface of the plate is coated with ink and, after wiping, the ink remains in the incised markings. The printing press consists of two assembled steel cylinders, one atop the other. The inked copper plate is then put on a steel plate on which there is placed a wet piece of Arches paper and a felt cloth; all of which is then passed between the cylinders. The result gives a light relief of the incisions on the paper.

My main inspiration, however, is different. It is in mathematics that helped me discover the meaning of models shown at the Institute

Henri Poincaré, and it reminds me on the time of my youth, at the Palais de la Découverte (see the article by Francis Apéry). If I take each of my visits to our capital to make prints of the rich Parisian scenery, it is in Provence that I compose my engravings of mathematics. I will now give an overview on three themes, namely, minimal surfaces, closed surfaces without singularity, and bi-periodic functions.

## 2. Minimal Surfaces

I started by studying certain surfaces of 3rd and 4th degree, given by simple equations that I could solve easily. I executed then lines thanks to the Cavaliere perspective, drawing carefully each of their remarkable curves. When data processing appeared, I programmed myself in BASIC to compute these surfaces starting from their parametric equations – today, I use standard software solutions. It was enough for me to choose the contour and the visual angle of my surfaces. Starting from a screen printing, I obtained a copy of a basis model being used for the future engraving.

I was interested in a family of surfaces which, since the 19th century, bear the name of minimal surfaces. The German school was very active in this field. Weierstraß in 1866 derived a representation formula for this particular family of surfaces:

$$x = \int(1-\zeta^2)R(\zeta)d\zeta \quad y = \int i(1+\zeta^2)R(\zeta)d\zeta \quad z = \int 2\zeta R(\zeta)d\zeta \qquad (1)$$

R ($\zeta$) is a function of a complex variable $\zeta = u + i\, v$ (i is the square root of -1), which determines the specific minimal surface. Taking the real part of x, y, and z gives the coordinate functions of the surface in the Euclidean 3-space which we also denote with x, y z.

### 2.1 Enneper Surface

The first minimal surface I studied with these formulas is that of Enneper (1863) :

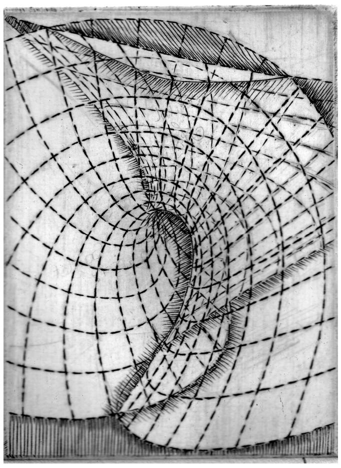

**Fig. 7. Enneper surface**

Here R ($\zeta$) = 3, from which one deduces, after integration

$$x = 3\zeta - \zeta^3 \quad y = i(3\zeta + \zeta^3) \quad z = 3\zeta^2.$$

Then after taking the real part we obtain

$$x = 3u + 3uv^2 - u^3,\ y = -(3v + 3u^2v - v^3),\ z = 3u^2 - 3v^2.$$

**2.2 Formulas of Monge and Weierstraß**

Thereafter, I used a generalized formula to represent minimal surfaces ($H^2=\zeta^2R$ and $F^2=R$):

$$x = \int(F^2 - H^2)d(\zeta) \quad y = \int i(F^2 + H^2)d(\zeta) \quad z = \int 2FHd(\zeta) \qquad (1')$$

All these formulas make it possible to compute minimal surfaces starting from an isometric network. The network traced on a surface depends on function "$\zeta$". :

- If one takes $\zeta = u + iv$, the conformal representation on the plane (since complex functions preserve angles) maps to a square-like grid on the minimal surface.
- To have a surface bordered by one or more closed curves, one will take $e^\zeta = e^u (\cos v + i \sin v)$, the conformal representation, here, will be made of concentric circles and radiant lines.

One can also use Monge's formula that one can, for example, deduce from the Weierstraß equations (1) or (1 '):

$$dx^2 + dy^2 + dz^2 = 0.$$

This formula is in particular satisfied by the following functions:

$$x = f(\zeta), \quad y = g(\zeta), \quad z = \int i \sqrt{(f(\zeta)^2 + g(\zeta)^2)}\, d\zeta.$$

The data of the functions F and G implies obviously x and y, and in general with some difficulty z, since computing z requires taking a square root. Since f and g are functions of a complex variable $\zeta$, they define points (x, y) of this plane along a curve. The writing of z, where f and g appear by their square, implies the presence of symmetry at least.

**2.3 Catalan Surface**

Here is an example established from the cycloid curve in the real plane, which is studied since 1501 (cf: http://www.mathcurve.com/courbes2d/cycloid/cycloid.shtml). The formula of the cycloid curve is:

$$x = u - \sin u, \quad y = \cos u$$

After replacement of u by $\zeta$, one obtains the value of z easily:

$$x = \zeta - \sin \zeta$$
$$y = \cos \zeta$$
$$z = 4i \sin \zeta/2$$

The developed equation gives us:

$$x = u - \sin u \cosh v$$
$$y = \cos u \cosh v$$
$$z = 4 \sin u/2 \sinh v/2$$

The parameters u and v being variable, we obtain the minimal surface of Catalan (1814-1894):

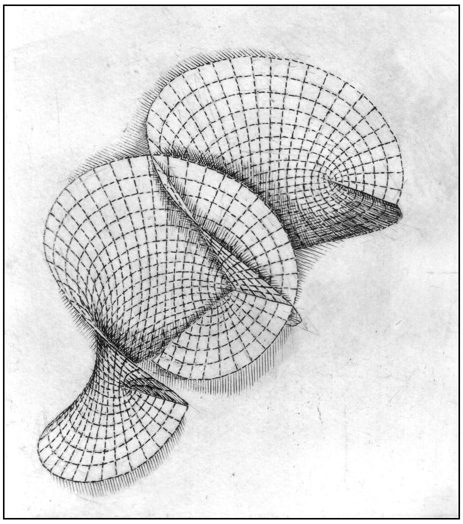

**Fig. 8. Catalan's minimal surface**

## 2.4 Jeener Surface

There exists many other simply plane curves from which one can obtain minimal surfaces. For example, starting from the planar spiral equation:

$$x = e^{mt} \cos t, \quad y = e^{mt} \sin t$$

one builds the spiral minimal surface:

$$x = e^{mt} \cos t$$
$$y = e^{mt} \sin t$$
$$z = i \sqrt{(1 + m^2)/m}$$

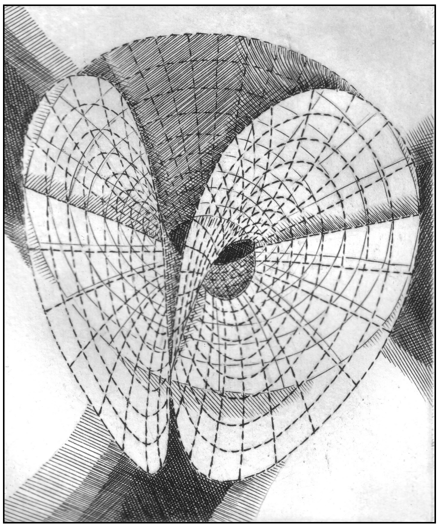

**Fig. 9. Minimal surface «à la Chouette»**

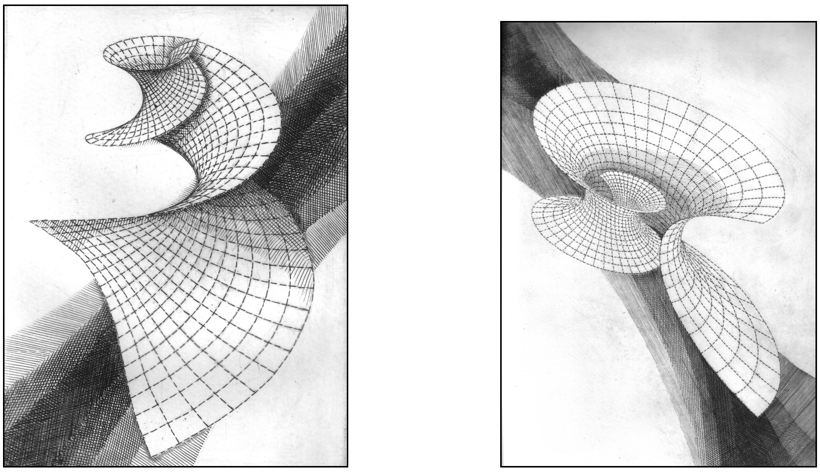

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**Fig. 10. Spiral minimal surfaces**

If, initially, I chose functions allowing me to trace traditional remarkable surfaces, I then left free course to my imagination. Here is an example:

$$x = u^m \cos mv - (m/(m + 2n)\ u^{m+2n}) \cos(m+ 2n)v$$
$$y = u^m \sin mv + (m/(m + 2n)\ u^{m+2n})\sin(m + 2n)v$$
$$z = (2mu^{m+n} /(m + n))\cos(m + n)v$$

These surfaces resemble flowers determined by constants m and n.

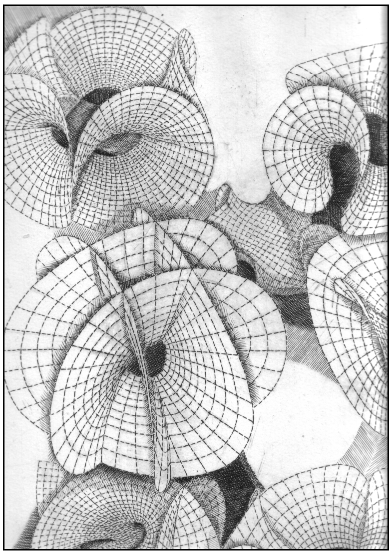

**Fig. 11. Floraison**

## 2.5 Minimal Surface with a Family of Parabolas

This surface, which comprises a family of parabolas, was studied by Enneper in 1882. It is determined by:

$$R(\zeta) = ia(\zeta^2 - 1)/\zeta^3 - i\ b/2\zeta^2$$

The equation is thus:

$$x = a\,u - a \sin u \,\mathrm{ch}\, v + b \sin u/2 \,\mathrm{sh}\, v/2$$
$$y = a - a \cos u \,\mathrm{ch}\, v + b \cos u/2 \,\mathrm{sh}\, v/2$$
$$z = 4\, a \sin u/2 \,\mathrm{sh}\, v/2 - b\, u/2$$

While making a = 1 and b = 0 one finds Catalan's surface, and for a = 0 and b = 1 the helicoid with planar axis.

**2.6 Bonnet Surface**

Within the framework of the general research on minimal surfaces having planar principal lines of curvature, Ossian Bonnet discovers a surface whose equation is:

$$x = u \cos m + \sin u \,\mathrm{ch}\, v$$
$$y = \sin m \cos u \,\mathrm{ch}\, v$$
$$z = v - \cos m \cos u \,\mathrm{ch}\, v$$

If one takes m = $\pi/2$, one finds the catenoid.

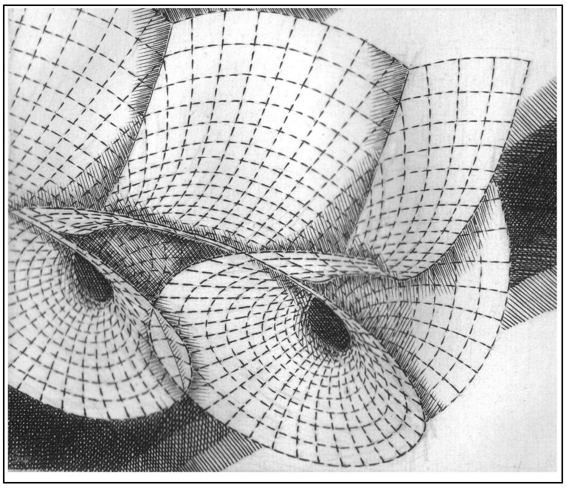

**Fig. 12. Bonnet minimal surface**

### 2.7 Henneberg Surface

Henneberg discovers the first one-sided resp. unilateral minimal surface, whose equation is:

$$x = 3 \cos u \operatorname{sh} v - \cos 3u \operatorname{sh} 3v$$

$$y = 3 \sin u \operatorname{sh} v + \sin 3u \operatorname{sh} 3v$$

$$z = 3 \cos 2u \operatorname{ch} 2v$$

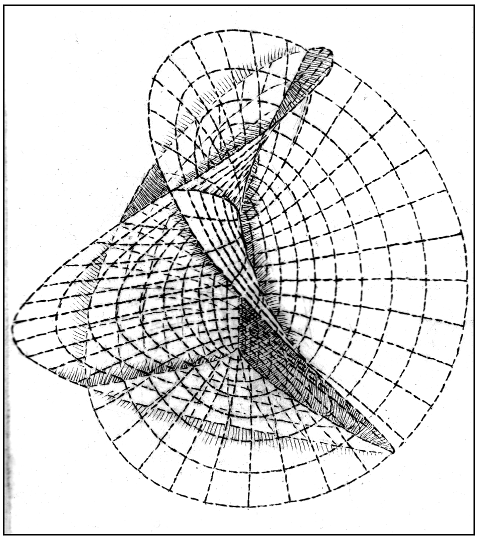

**Fig. 13. Henneberg surface**

## 3. Topology of Closed Surfaces without Singularities

3.1 The first closed unilateral surface, the famous Klein bottle, was discovered by Felix Klein in 1882. The Klein bottle can be generalized to a surface having "n" bottles suing the following general equation:

$$W = \cos((m+1)u+\pi/(m+1))+3/2$$
$$x = m \cos u + \cos mu - (m+1)/2m\ W \sin (m-1)u/2 \cos v$$
$$y = m \sin u - \sin mu - (m+1)/2m\ W \cos (m-1)u/2 \cos v$$
$$z = W \sin v$$

These surfaces are generated by a family of circles whose centres move on a hypocycloid with n cusps. The surface is unilateral when the number of cusps is odd: $n = 2m + 1$.

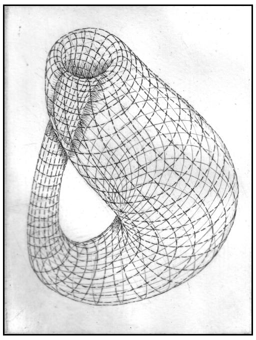

**Fig. 14. Klein-bottle**

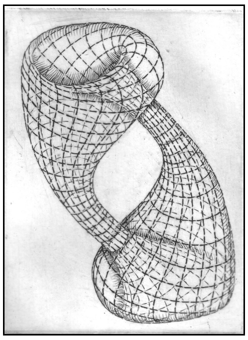

**Fig. 15. Double Klein-bottle**

The simplest generalized bottle is the triple bottle ($m = 2$, $N = 3$):

$$W = \cos(3u + \pi/4) + 3/2$$
$$x = 2\cos u + \cos 2u - 3/4W \sin u/2 \cos v$$
$$y = 2\sin u - \sin 2u - 3/4W \cos u/2 \cos v$$
$$z = W \sin v$$

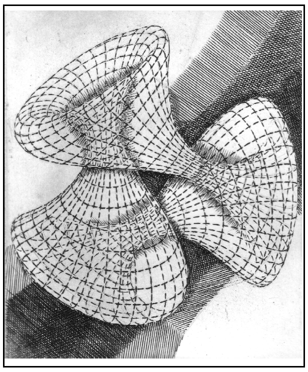

**Fig. 16. Triple Klein-Jeener bottle**

**3.2 The First Unilateral Closed Surface** with only one pole was discovered by Werner Boy in 1902. It has a curve of self-intersection, a three-bladed propeller. There exists a family of these surfaces. They have an odd symmetry. In the case, for example, where symmetry is of order 5, the lines of self-intersection are made up of two propellers with 5 blades of different sizes.

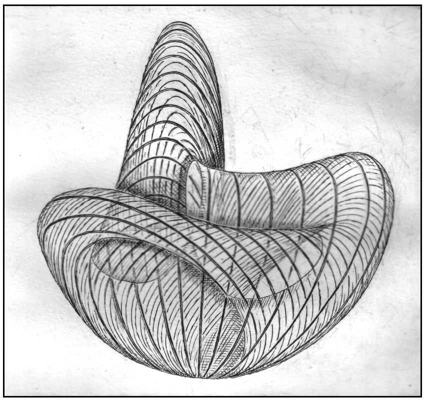

**Fig. 17. Boy surface with symmetry of order 3**

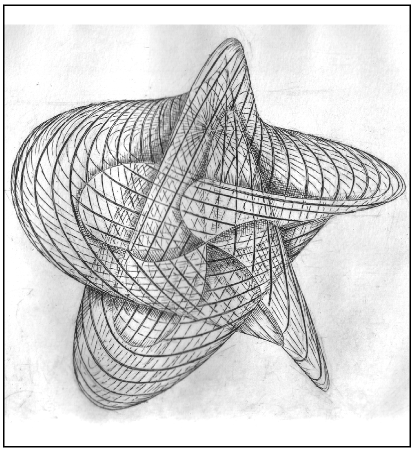

**Fig. 18 Boy surface with symmetry of order 5**

There is a connected family, of even symmetry. Surfaces of the two families can be used, in an essential way, to evert the sphere. Bernard Morin introduced and used that surface whose symmetry is of order 4 and which bears his name. The eversion occurs at a central stage where two models exist: the open one and the closed. The last model has a curve of self-intersection, a four-bladed propeller and two circles. (cf. the article of Richard Denner in this volume). Surfaces of symmetry of order 3 and 4 have a common point: can be traced on each one of them and entirely recover a set of ellipses. Their equations were published by François Apéry [1].

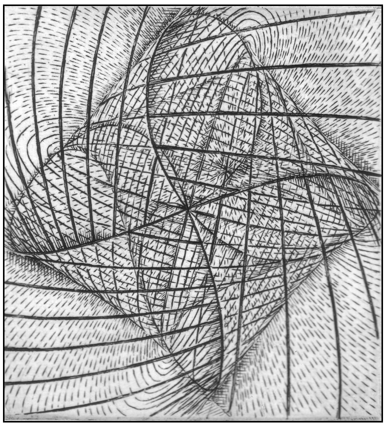

Fig. 19 Apéry model

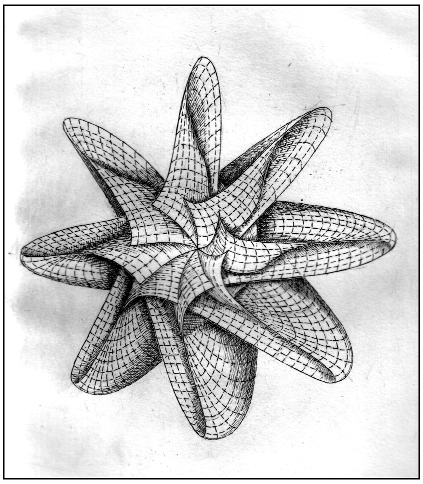

Fig. 20. Morin surface of order 8

These surfaces, thanks to their graphic complexity, emphasize the technique of engraving: indeed, one combines, here, the rigour of a basic data-processing layout with that of a work hand made. The thickness and the texture of the curves, and in certain cases, of the lines of self-intersection, must allow a good legibility of surface.

### 3.3 Surfaces with Constant Total Curvature.

Each point on a surface is the intersection of a pair of two orthogonal lines of curvature. The curvature of the surface at this point is the product of the curvatures of the two lines of curvature. When this local curvature is same for each point of the surface then it is said that surface has constant curvature K. Surfaces of this type are the sphere (K = 1), the plane (K = 0), and the pseudo-sphere (K = -1). Surfaces of negative curvature are called *hyperbolic*.

#### 3.3.1 The Pseudo-sphere (K = -1)

Studied in particular by the Italian mathematician Beltrami in 1868, the pseudo-sphere is a surface of revolution generated by the tractrix, a curve introduced about 1670 per Claude Perrault (cf. http://www.mathcurve.com/courbes2d/tractrice/tractrice.shtml).
Here is the equation of the pseudo-sphere:

x = cos u / cosh v
y = sin u /cosh v
z = v - tanh v

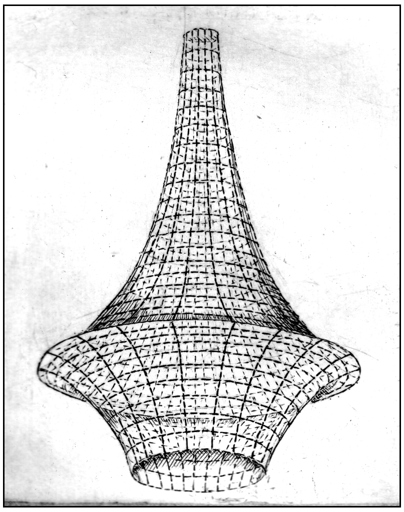

**Fig. 21. Pseudo-sphere**

Here two other examples where K is equal to a negative constant: Dini surface, a helicoid, and Kuen surface, a 2-soliton.

**3.3.2 Dini Helicoid (K = -1)**

This helicoid is also generated by a tractrix; the second family of curves is formed, here, of circular propellers. The equation can thus take the following form:

x = cos u /cosh v
y = sin u/ cosh v
z = v-tanh v +u/4

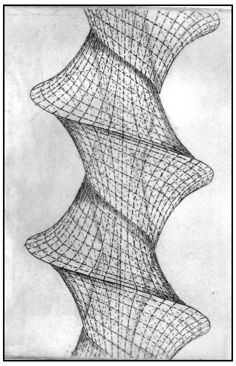

**Fig. 22. Dini surface**

### 3.3.3 Kuen Surface (K = -1)

The richness of the surface of Kuen allured several artists. One can see a beautiful model at the Institute Henri Poincaré. It inspired a picture by Luc Bénard (cf. the article of Richard Palais in this volume). Here is the equation:

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$z = \log \tan v/2 + a \cos \theta$$

with $\theta = u - \arctan u$, $a = 2/(1 + u^2 \sin^2 v)$, and $r = a \sqrt{(1 + u^2)} \sin v$

and the engraving I made:

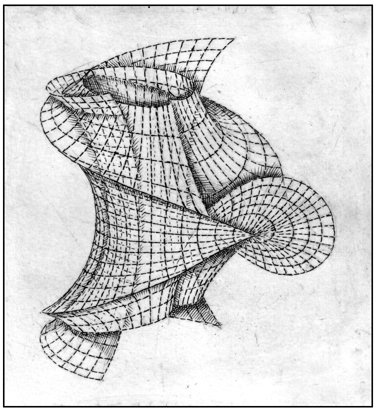

**Fig. 23. Kuen surface**

### 3.3.4 Sievert Surface (K = + 1)

The Sievert surface is an example with constant positive curvature K = 1. Its equation is:

$$R = 2/\sqrt{(3)}*(2\sin v\ \sqrt{(1+3\sin^2 u)})/(4-3\sin^2 v\ \cos^2 u)$$
$$\theta = -u/2 + \arctan(2 + \tan u)$$
$$x = R \cos \theta$$
$$y= R \sin\ \theta$$
$$z = 1/\sqrt{(3)}*[\log (\tan (t/2))+8 \cos t/(4-3 \sin^2 t \cos^2 u)]$$

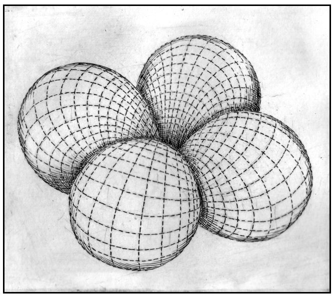

**Fig. 24. Sievert's surface**

### 3.3.5 Surfaces of Zero Curvature (K = 0)

Surfaces with K = 0 are locally isometric to a plane. Except the cones and the cylinders, these surfaces are the loci of the tangents to a given skew curve.

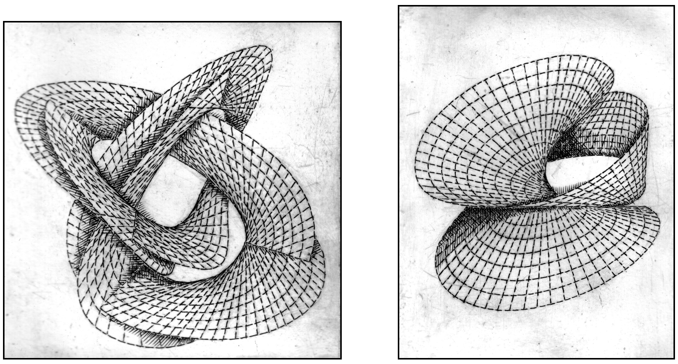

**Fig. 25. et 26. Skew surfaces**

## 4. Bi-periodical Functions

### 4.1 Jacobi Functions

Among the complex functions, the first function with two periods was discovered on the basis of the following elliptic integral:

$$F=\int dx/\sqrt{(1-x^2)(1-k^2x^2)}.$$

By inverting this integral, one obtains the Jacobi function called sine amplitude, sn u. There exist two other functions of Jacobi: cosine amplitude, cn u, and delta amplitude, dnu. These functions are connected in the following way: cn u $=\sqrt{(1-sn^2u)}$ et dn u $=\sqrt{(1-k^2sn^2u)}$.

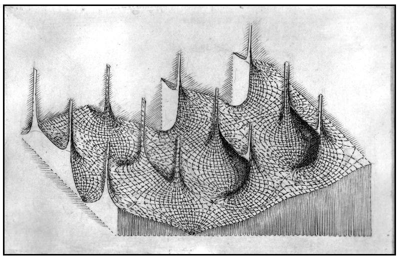

**Fig. 27. Amplitude function**

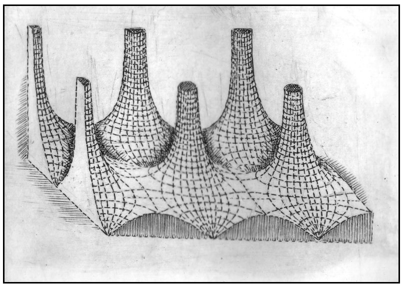

**Fig. 28. Sinus-amplitude function**

### 4.2 Weierstraß Functions

Weierstraß studies, in its turn, the bi-periodical functions which bear his name. One starts from the integral $u = \int ds/\sqrt{(4s^3- g_2 s- g_3)}$ where $g_2$ and $g_3$ are invariants. $e_1$, $e_2$ and $e_3$ are the roots of the equation $4s^3- g_2 s- g_3 = 0$.

These functions are denoted by $\wp$ and $\wp'$, where $\wp'$ is the derivative of $\wp$. One thus gets for the first three terms of the series:

$$\wp = 1/u^2 + g_2 u^2/20 + g_3 u^4/28$$
$$\wp' = -2/u^3 + g_2 u/10 + g_3 u^3/7\ldots.$$

For the engraving of the function $\wp$, one takes $g_2 = 4$ and $g_3 = 0$ (case of the lemniscate) which gives us:

$$e_1 = -1,\ e_2 = 0 \text{ and } e_3 = 1.$$

The two periods are, here, equal, and their value is 2.622…

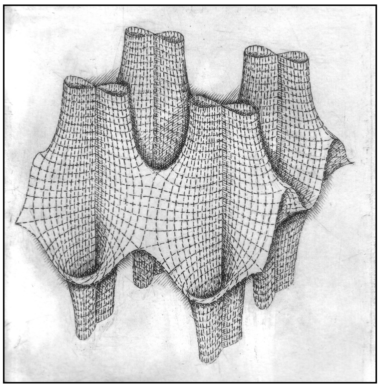

**Fig. 29. Weierstraß function $\wp$**

As regard of the function $\wp$ ', we choose the equi-anharmonic case $g_2 = 0$ and $g_3 = 1$. Engraving shows well that a section by a plane close to the top cuts the surface according to curves in the clover shape. The plane $z = 0$ cuts the surface according to a tessellation by equilateral triangles.

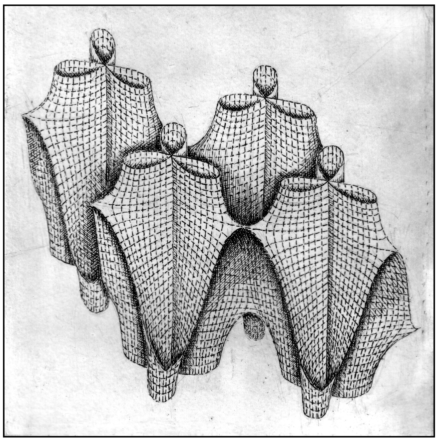

**Fig. 30. Weierstraß function $\wp'$**

## References

I am currently carrying out the project of engraving an atlas of mathematical models. Here is a list of books which have particularly influenced for my study of surfaces:

[1] Apéry, F., *Models of the Real Projective Plane*, Vieweg, Braunschweig-Wiesbaden, 1987.
[2] Darboux, G., *Théorie Générale des Surfaces*, Chelsea, New-York, 1986.
[3] Fischer, G., *Mathematical Models*, Vieweg, Braunschweig-Wiesbaden, 1986.
[4] Jahnke and Emde, *Tables of Functions*, Dover, New-York, 1945.
[5] Nitsche, J.C.C., *Lectures of Minimal Surfaces*, Cambridge, 1989.

I also took as a starting point the collection of plaster models published by Martin Schilling (Leipzig 1911). One can see, in France, some of these models at the Institute Henri Poincaré and at the Palais de la Découverte.

I would like to thank Claude Bruter, Richard Palais, Simon Salamon and in particular Konrad Polthier for their help in the linguistic preparation of the manuscript.